Published by Institute of Physics Publishing for SISSA/ISAS
Received: *February 11, 2002*
Accepted: *March 06, 2002*

I. Non-commutativity and supersymmetry

Sonia Paban
Department of Physics, University of Texas
Austin, TX 78712, USA
E-mail: sethi@theory.uchicago.edu

Savdeep Sethi
Enrico Fermi Institute, University of Chicago
Chicago, IL 60637, USA, and
School of Natural Sciences, Institute for Advanced Study
Princeton, NJ 08540, USA
E-mail: sethi@theory.uchicago.edu

Mark Stern
Department of Mathematics, Duke University
Durham, NC 27708, USA
E-mail: stern@math.duke.edu

Abstract: We study the extent to which the gauge symmetry of abelian Yang-Mills can be deformed under two conditions: first, that the deformation depend on a two-form scale. Second, that the deformation preserve supersymmetry. We show that (up to a single parameter) the only allowed deformation is the one determined by the star product. We then consider the supersymmetry algebra satisfied by NCYM expressed in commutative variables. The algebra is peculiar since the supercharges are not gauge-invariant. However, the action, expressed in commutative variables, appears to be quadratic in fermions to all orders in θ.

Keywords: Renormalization Regularization and Renormalons, Gauge Symmetry, Extended Supersymmetry, Non-Commutative Geometry.

JHEP03(2002)012

Contents

1. Introduction

The appearance in string theory of non-commutative Yang-Mills [1] leads us to a natural question: how unique is the deformation that leads from commutative to non-commutative Yang-Mills? It is natural to consider the question in the context of supersymmetric theories since they are most likely to exist at the quantum level. The cleanest example is $N = 1$ Yang-Mills in ten dimensions. The goal of this work is to ask how we can deform this SYM (supersymmetric Yang-Mills) theory while still preserving supersymmetry. In ten dimensions, the gauge coupling constant, g^2, is dimensionful. However, we will not consider deformations which depend on higher powers of the coupling constant. Rather, we introduce a two-form scale θ_{ij} with mass dimension -2 and consider deformations that depend on this scale. For simplicity, we also restrict to deformations of abelian Yang-Mills. Our goal is to understand to what extent the non-commutative generalization is determined by supersymmetry. For a review of earlier work on non-commutative field theory, see [2].

In section 2, we begin by recalling the structure of abelian SYM. We then proceed to describe the form and constraints on deformations compatible with supersymmetry. The deformations we permit are quite general, and can, in principle, look quite bizarre. For example, fermion bilinears can appear in the deformations. We also make no special assumptions about the structure of the deformation (like associativity of a product). In section 3, we undertake the task of unraveling the constraints. To $\mathrm{O}(\theta)$, we show that the unique deformation of the gauge symmetry compatible with supersymmetry is precisely the deformation given by the star product commutator,

$$\begin{aligned}\delta_\Lambda A_\mu &= \Lambda_{,\mu} + \theta^{ij}\left(\partial_i\Lambda\partial_j A_\mu - \partial_i A_\mu \partial_j \Lambda\right) \\ &= \Lambda_{,\mu} + f_1(\Lambda, A_\mu)\,, \end{aligned} \tag{1.1}$$

where δ_Λ generates a gauge transformation. This is a closed algebra at $\mathrm{O}(\theta)$ and no terms with higher powers of θ are required. There is also no free parameter that cannot be absorbed by rescaling θ.

Beyond $\mathrm{O}(\theta)$, it makes the analysis considerably easier to impose the condition that the supercharges, δ_ϵ, be gauge-invariant,

$$[\delta_\Lambda, \delta_\epsilon] = 0\,, \tag{1.2}$$

as an off-shell condition. In both unperturbed abelian and non-abelian SYM theories, this condition indeed holds off-shell without use of the equations of motion. Under this restriction, we show that by the use of field redefinitions, a gauge transformation can always be brought into the form

$$\delta_\Lambda A_\mu = \Lambda_{,\mu} + f_1(\Lambda, A_\mu) + f_3(\Lambda, A_\mu) + \sum_{p>3} f_p(\Lambda, A_\mu)\,, \tag{1.3}$$

where f_i is an $\mathrm{O}(\theta^i)$ bilinear. An analogous statement holds for δ_Λ acting on a fermion. Each f_i is completely determined, and agrees with the form predicted from the star product commutator. The overall coefficient of each f_i for $i > 3$ is determined relative to the coefficient of f_3. The only freedom is the overall coefficient of f_3. This is the only free parameter permitted in the deformation.

We expect our argument to descend with little modification to lower dimensional theories. In these theories (for example, $N = 4$ $D = 4$ SYM), it then appears that proving perturbative renormalizability reduces to showing that this one coefficient is invariant under renormalization group flow. We should also point out that while supersymmetry is a key ingredient in our argument, maximal supersymmetry is not. It would be interesting to see how few supercharges are needed to guarantee the star product structure. Another direction worth mentioning is the extension of these results to the effective action of $N = 4$ SYM. Combining this analysis with the techniques in [3] should lead to a non-perturbative way of obtaining gauge-invariant F^4 and F^6 interactions, extending the work of [4, 5, 6].

In this analysis, we do not constrain higher derivative gauge invariant deformations of the action. From the work of [7, 8] on the DBI action and supersymmetry, and the explicit supersymmetrization of DBI given in [9], we certainly expect constraints on higher derivative interactions.

In section 4, we turn our attention to a puzzle. In [10], it is argued that the noncommutative theory can be rewritten in terms of commutative variables. From the commutative perspective, the action contains an infinite string of irrelevant interactions. The structure of these interactions does not seem particularly remarkable. Yet if the deformation of the gauge symmetry is so constrained by supersymmetry, we should ask: how do these constraints appear in terms of commutative variables? What we argue is that the usual supersymmetry algebra is modified in a peculiar way. In particular, the supercharges, δ_ϵ, are no longer gauge-invariant but rather

$$[\delta_\Lambda, \delta_\epsilon] = \delta_{\Lambda'} \,. \tag{1.4}$$

This is reminiscent of the structure that appears in a supergravity theory, although supersymmetry still seems to be a global symmetry. The second peculiarity is that the action need not contain couplings that are more than quadratic in the fermions, i.e., the simplest extension of the map of [10] to fermions leads only to quadratic couplings in the action. This is quite different from what we might have expected in a theory with an infinite number of higher derivative terms. The supersymmetry algebra is correspondingly highly constrained. To what extent these structures generalize to a $D = 6$ chiral tensor multiplet is a topic of current investigation [11].

2. $N = 1$ $D = 10$ SYM

2.1 The undeformed abelian theory

Let us begin by reviewing the structure of abelian $N = 1$ SYM, and establishing our notation. We choose a spacetime metric $\eta_{\mu\nu} = (-, +, \dots, +)$. Our only symmetry group is the Lorentz group $\mathrm{SO}(9,1)$. In terms of real gamma matrices Γ_μ , $\mu = 0, \dots, 9$ satisfying,

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\,\eta_{\mu\nu}\,, \tag{2.1}$$

we can express the supersymmetry variation of the gauge-field A_μ and real fermion Ψ:

$$\begin{aligned} \delta_\epsilon A_\mu &= \frac{i}{2}\,\overline{\epsilon}\,\Gamma_\mu \Psi\,, \\ \delta_\epsilon \Psi &= -\frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon\,. \end{aligned} \tag{2.2}$$

We define $\overline{\epsilon} = \epsilon^T \Gamma^0$ as usual, and the fermion is Weyl so there are 16 real components. The dimension of A_μ is 1 while the dimension of Ψ is 3/2. Let us explicitly check closure of the supersymmetry algebra. The supersymmetry algebra only closes up to gauge transformations, which we denote by δ_Λ. The algebra must satisfy,

$$\begin{aligned} [\delta_\Lambda, \delta_\epsilon] &= 0\,, \\ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= \frac{i}{2}\overline{\epsilon}_2 \Gamma^\mu \epsilon_1 \partial_\mu + \delta_\Lambda\,. \end{aligned} \tag{2.3}$$

The first condition is the statement that there is a well-defined global supercharge, independent of any gauge choice. The second condition is simply the statement that we have a closed symmetry algebra. Note, however, that closure is on Poincaré together with a gauge transformation. This does not make geometric sense for non-commutative (or non-abelian) theories since we expect closure on D_μ rather than ∂_μ. As we shall see, the algebra does close on the geometrically sensible covariant derivative. This occurs because the gauge parameter, Λ, appearing on the right-hand side of (2.3) will actually become field-dependent. Lastly, we note that the gauge transformations themselves must leave invariant the equations of motion, and form a closed algebra.

On the gauge-fields, we see that

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A_\mu &= \frac{i}{2}\bar{\epsilon}_2 \Gamma_\nu \epsilon_1 \partial^\nu A_\mu - \frac{i}{2}\partial_\mu(\bar{\epsilon}_2 \Gamma_\nu A^\nu \epsilon_1) \\
&= \frac{i}{2}\bar{\epsilon}_2 \Gamma^\nu \epsilon_1 F_{\nu\mu} \, .
\end{aligned} \tag{2.4}
$$

For this to satisfy (2.3), we see that

$$
\delta_\Lambda A_\mu = -\frac{i}{2}\partial_\mu(\bar{\epsilon}_2 \Gamma_\nu A^\nu \epsilon_1) \tag{2.5}
$$

for the undeformed theory. This is a conventional gauge transformation but the gauge parameter is field-dependent as promised. Note that the right-hand side of (2.4) depends only on the gauge-invariant combination, $F_{\mu\nu}$. This is worth noting because when we come to the five-brane, the correct gauge-invariant objects to study are unknown.

Closure of the supersymmetry algebra on Ψ gives,

$$
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \Psi = \frac{i}{2}\bar{\epsilon}_1 \Gamma^\mu \epsilon_2 \partial_\mu \Psi + \cdots \, . \tag{2.6}
$$

The omitted terms are all proportional to the equation of motion for Ψ so, as usual, the algebra only closes on-shell.

2.2 The deformed abelian theory

How can we deform the supersymmetry transformations? The most general deformation can be captured by the following modifications,

$$
\begin{aligned}
\delta_\epsilon A_\mu &= \frac{i}{2}\,\bar{\epsilon}\,\Gamma_\mu \Psi + \bar{\epsilon}\, N_\mu \Psi \, , \\
\delta_\epsilon \Psi &= -\frac{1}{4} F_{\mu\nu}\Gamma^{\mu\nu}\epsilon + M\epsilon \, ,
\end{aligned} \tag{2.7}
$$

where N and M are unspecified operators. It will often be convenient to rewrite these transformations without explicit epsilons,

$$
\begin{aligned}
\delta_a A_\mu &= \frac{i}{2}\,(\Gamma^0\Gamma_\mu \Psi)_a + (\Gamma^0 N_\mu \Psi)_a \, , \\
\delta_a \Psi_b &= -\frac{1}{4} F_{\mu\nu}\Gamma^{\mu\nu}_{ba} + M_{ba} \, ,
\end{aligned} \tag{2.8}
$$

We do not want to expand N and M in derivatives, but we can classify terms in N or M by number of fermions. Each of these terms can have an arbitrary number of derivatives. The natural physical constraint is that N, M vanish when $\theta_{ij} \to 0$.

Field redefinitions play a crucial role in our analysis. For N_μ of the form

$$
N_\mu \sim \Gamma_\mu(\text{anything}) \, ,
$$

where (anything) is Lorentz invariant, we can make a field redefinition

$$
\widetilde{\Psi} = (1 + \text{anything})\Psi
$$

to remove this kind of N_μ. Unfortunately, not all N_μ are of this convenient form. We also need to keep in mind possible field redefinitions of the gauge-field,

$$\widetilde{A_\mu} = A_\mu + h_\mu(\theta, A, \Psi)\,,$$

where $h_\mu \to 0$ as $\theta \to 0$, and h_μ is gauge invariant. For recent comments on controlling field redefinitions in supersymmetric theories, see [12].

In studying non-trivial deformations, we must also consider the possibility that our gauge symmetry is modified. Let us express our modified gauge symmetry in the following way,

$$\begin{aligned} \delta_\Lambda\, A_\mu &= \Lambda_{\,,\mu} + B^\Lambda_\mu\,, \\ \delta_\Lambda\, \Psi_a &= C^\Lambda_a\,, \end{aligned} \tag{2.9}$$

where B and C are at least of order θ. Where convenient, we will express the modified gauge transformation in a power series in θ,

$$\delta_\Lambda = \delta^0_\Lambda + \delta^1_\Lambda + \delta^2_\Lambda + \cdots\,,$$

where δ^n_Λ is $\mathrm{O}(\theta^n)$. Also note that,

$$[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{\Lambda_\theta}\,, \tag{2.10}$$

where Λ_θ is at least $\mathrm{O}(\theta)$ because we started with an abelian theory so the $\mathrm{O}(0)$ terms commute.

2.3 The general form of the closure constraints

We start with the relation,

$$\delta_{\Lambda_1} + \delta_{\Lambda_2} = \delta_{\Lambda_1+\Lambda_2}\,. \tag{2.11}$$

This is forced by considering the lowest unperturbed gauge transformations. However, this constraint implies that δ_Λ is linear in the gauge parameter Λ. We still require that the supercharges be well-defined with respect to the gauge symmetry so,

$$[\delta_\Lambda, \delta_a] = 0\,. \tag{2.12}$$

In this, and all subsequent discussion, we must keep in mind that the "0" appearing on the right-hand side could include terms that vanish on-shell. If (2.12) is not true then the supercharges are not gauge-invariant. In most cases, this implies that supersymmetry is part of the gauge symmetry, and we are forced to consider a theory of supergravity. We will therefore impose this condition. Let us apply this constraint first to the gauge-field,

$$\begin{aligned} [\delta_\Lambda, \delta_a] A_\mu &= \delta_\Lambda \left(\Gamma^0 \left\{ \frac{i}{2}\Gamma_\mu \Psi + N_\mu \Psi \right\} \right)_a - \delta_a B^\Lambda_\mu \\ &= \frac{i}{2}(\Gamma^0\Gamma_\mu C^\Lambda)_a + \delta_\Lambda(\Gamma^0 N_\mu \Psi)_a - \delta_a B^\Lambda_\mu \\ &= 0\,. \end{aligned} \tag{2.13}$$

This gives us the following relation for C in terms of B:

$$C_a^\Lambda = 2i(\Gamma^\mu \Gamma^0\, \delta)_a B_\mu^\Lambda - 2i\, \delta_\Lambda (\Gamma^\mu N_\mu \Psi)_a\,. \tag{2.14}$$

Note that we do *not* have to sum on μ in (2.14). However, the expression for C^Λ must be independent of μ.

Applying (2.12) to the fermion gives the following relation,

$$[\delta_\Lambda, \delta_a]\Psi_b = \delta_\Lambda \left(-\frac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu} + M\right)_{ba} - \delta_a C_b^\Lambda = 0\,. \tag{2.15}$$

We now turn to closure of the gauge algebra. Acting on the fermion Ψ, we obtain the following relations

$$\begin{aligned}
[\delta_{\Lambda'}, \delta_\Lambda]\Psi_a &= \delta_{\Lambda'} C_a^\Lambda - \delta_\Lambda C_a^{\Lambda'} \\
&= \frac{\delta C_a^\Lambda}{\delta A_\nu}(\Lambda'_{,\nu} + B_\nu^{\Lambda'}) - \frac{\delta C_a^{\Lambda'}}{\delta A_\nu}(\Lambda_{,\nu} + B_\nu^{\Lambda}) + \frac{\delta C_a^\Lambda}{\delta \Psi_b}(C_b^{\Lambda'}) - \frac{\delta C_a^{\Lambda'}}{\delta \Psi_b}(C_b^{\Lambda}) \\
&= \delta_{\Lambda_\theta}\Psi = C_a^{\Lambda_\theta}\,.
\end{aligned} \tag{2.16}$$

Note that the functional derivatives $\frac{\delta C_a^\Lambda}{\delta A_\nu}(\cdots)$ and $\frac{\delta C_a^\Lambda}{\delta \Psi_b}(\cdots)$ are operators that act on their respective arguments. Dimensional analysis is kind to us here. The term $C_a^{\Lambda_\theta}$ is at least $\mathrm{O}(\theta^2)$. In a θ expansion, the dominant term in $C_a^{\Lambda_\theta}$ is,

$$\frac{\delta C_a^\Lambda}{\delta A_\nu}(\Lambda'_{,\nu}) - \frac{\delta C_a^{\Lambda'}}{\delta A_\nu}(\Lambda_{,\nu})\,. \tag{2.17}$$

This term is only $\mathrm{O}(\theta)$ and must vanish.

Closure of the gauge algebra on the gauge-fields gives us the relation,

$$\begin{aligned}
[\delta_{\Lambda'}, \delta_\Lambda]A_\mu &= \delta_{\Lambda'} B_\mu^\Lambda - \delta_\Lambda B_\mu^{\Lambda'} \\
&= \frac{\delta B_\mu^\Lambda}{\delta A_\nu}(\Lambda'_{,\nu} + B_\nu^{\Lambda'}) - \frac{\delta B_\mu^{\Lambda'}}{\delta A_\nu}(\Lambda_{,\nu} + B_\nu^{\Lambda}) + \frac{\delta B_\mu^\Lambda}{\delta \Psi_b}(C_b^{\Lambda'}) - \frac{\delta B_\mu^{\Lambda'}}{\delta \Psi_b}(C_b^{\Lambda}) \\
&= \delta_{\Lambda_\theta} A_\mu = \Lambda_{\theta,\mu} + B_\mu^{\Lambda_\theta}\,.
\end{aligned} \tag{2.18}$$

There are further conditions from closure of the supersymmetry algebra which we shall describe when needed.

3. Unraveling the constraints

3.1 The gauge parameter Λ_θ

Our first step is to learn something about the gauge parameter Λ_θ that appears in the commutator,

$$[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{\Lambda_\theta}\,,$$

where Λ, Λ' are field-independent. We know that Λ_θ is an antisymmetric bilinear form in Λ and Λ'. What we wish to show is that Λ_θ is field-independent. Using (2.12), it is easy to show that

$$[\delta_a, \delta_{\Lambda_\theta}] = 0\,, \tag{3.1}$$

up to terms that vanish on-shell. The on-shell caveat is needed because the SUSY algebra typically only closes on-shell. The parameter which determines the deformation of our gauge symmetry therefore commutes with the SUSY vector-fields, δ_a.

If we apply (3.1) to A_μ, we learn that

$$\begin{aligned}[\delta_a, \delta_{\Lambda_\theta}]A_\mu &= \delta_a(\Lambda_{\theta,\mu} + B_\mu^{\Lambda_\theta}) - \delta_{\Lambda_\theta}\left(\frac{i}{2}(\Gamma^0\Gamma_\mu\Psi)_a + (\Gamma^0 N_\mu\Psi)_a\right) \\ &= \delta_a^0(\Lambda_{\theta,\mu}) + \mathrm{O}(\theta^2) \\ &= 0\,. \end{aligned} \tag{3.2}$$

At $\mathrm{O}(\theta)$, there is no freedom to construct a term in Λ_θ from fermions on dimension grounds. Any term constructed from bosons must vanish by (3.2). This statement is true on the nose rather than just on-shell: the only term that could vary into an equation of motion has the form,

$$\partial^\nu A_\nu \theta\Lambda\Lambda'.$$

This is neither antisymmetric in Λ, Λ' nor is there a way to contract the indices. So Λ_θ is field independent to $\mathrm{O}(\theta)$.

To go beyond $\mathrm{O}(\theta)$, we note that (3.2) is satisfied for any Λ_θ independent of A, Ψ. This teaches us that,

$$\delta_a(\Lambda_{\theta,\mu}) + B_\mu^{\delta_a\Lambda_\theta} = 0\,.$$

Now this is a trianglar set of equations which reduce to the statement

$$\delta_a^0(\Lambda_\theta) = 0\,. \tag{3.3}$$

If we apply a second δ_a^0 and use closure of the supersymmetry algebra at lowest order, we see that

$$\delta_a^0\delta_a^0(\Lambda_\theta) \sim (\Gamma^\nu_{aa}\partial_\nu + \delta^0_{\Gamma^\nu_{aa}A_\nu})\Lambda_\theta = 0\,.$$

We can then conclude that Λ_θ is independent of fermions (on which δ_Λ^0 vanishes). Returning to (3.3) then teaches us that Λ_θ is independent of bosons. We therefore conclude that Λ_θ is field independent to all orders in θ up to terms whose variation is zero on-shell. We now impose (2.12) as an off-shell constraint. Under this stronger constraint, Λ_θ must be field independent to all orders in θ.

3.2 The Jacobi identity

Now that we have established that Λ_θ is field independent, we can try to construct Λ_θ in terms of $\Lambda, \Lambda', \theta$ and derivatives. Let us write the gauge parameter Λ_θ in terms of a function f,

$$\Lambda_\theta = f(\Lambda, \Lambda')\,, \tag{3.4}$$

where f is antisymmetric in its two arguments. To order θ, it is easy to see that there is a unique possibility,

$$f(\Lambda, \Lambda') = \theta^{ij}(\partial_i\Lambda\partial_j\Lambda' - \partial_i\Lambda'\partial_j\Lambda) + \mathrm{O}(\theta^2)\,. \tag{3.5}$$

This leading term itself defines a closed supersymmetry algebra. Nevertheless, we can ask about the structure of possible higher θ corrections to f.

JHEP03(2002)012

In performing this analysis, it is rather crucial that the Jacobi identity is obeyed by f. Since we deal with standard operator composition, our gauge transformations automatically satisfy the usual Jacobi identity

$$[\delta_\Lambda, [\delta_{\Lambda'}, \delta_{\Lambda''}]] + \text{cyclic} = 0\,.$$

In turn, this relation implies that

$$f(\Lambda, f(\Lambda', \Lambda'')) + f(\Lambda', f(\Lambda'', \Lambda)) + f(\Lambda'', f(\Lambda, \Lambda')) = 0\,. \tag{3.6}$$

3.3 Solving for Λ_θ.

At order θ, we introduce no free parameters. Any choice can be absorbed into a redefinition of θ. Beyond $\mathrm{O}(\theta)$, we will introduce parameters. The key question in understanding the renormalizability and uniqueness of our deformation is, how many? An explicit computation shows that there are no $\mathrm{O}(\theta^2)$ deformations of the gauge algebra consistent with (3.6). The first non-trivial deformation is $\mathrm{O}(\theta^3)$.

In order to analyze the structure of allowed deformations, we expand f in a series,

$$f = f_1 + f_3 + \sum_{p>3} f_p\,, \tag{3.7}$$

where each f_k is homogeneous of order k in θ. Since θ has mass dimension -2, each f_k must be homogeneous of degree $2k$ in derivatives. Since f is antisymmetric, so too is each f_k. Since we will consider only antisymmetric functions, it is convenient to adopt exterior algebra notation. We therefore write,

$$f_k = m_k^{IJ} \partial_I \wedge \partial_J.$$

Here $I = \{i_1, \ldots, i_n\}$ and J are multi-indices. We define

$$|I| = i_1 + \cdots + i_n\,, \qquad \partial_I = \partial_1^{i_1} \cdots \partial_n^{i_n}\,,$$

and m_k^{IJ} is zero unless $|I|+|J| = 2k$. When I is empty, we set $\partial_I = 1$. The exterior algebra notation is to be interpreted as usual,

$$f_k(x,y) = m_k^{IJ} \partial_I(x) \wedge \partial_J(y) - m_k^{IJ} \partial_I(y) \wedge \partial_J(x)\,.$$

This allows us to view differential operators as differential forms. We will similarly define 3-form differential operators

$$\begin{aligned} \partial_I \wedge \partial_J \wedge \partial_K(x,y,z) \;=\; & \partial_I(x) \wedge \partial_J(y) \wedge \partial_K(z) + \partial_I(y) \wedge \partial_J(z) \wedge \partial_K(x) + \\ & + \partial_I(z) \wedge \partial_J(x) \wedge \partial_K(y)\,. \end{aligned} \tag{3.8}$$

The generalization to higher forms is straightforward.

We can now express the Jacobi identity in terms of our expanded f. It gives the equalities

$$\sum_{i+j=k+1} \left(f_i(f_j(x,y),z) + f_i(f_j(y,z),x) + f_i(f_j(z,x),y) \right) = 0\,. \tag{3.9}$$

We can view this equation as inductively defining f_k in terms of the f_j, $j < k$. To what extent are the f_j uniquely determined? We started with the abelian Lie algebra of functions. Turning on f_1 deforms this to the Poisson Lie algebra, which has 1 free parameter associated with scaling the symplectic form θ. For the Lie algebra determined by the $*$-product, this is the only free parameter.

However, since the $\mathrm{O}(\theta)$ deformation gives a Lie algebra without need for any higher terms, we also have a free parameter which corresponds to scaling f_3. What we will show is that there are no other free parameters. If the coefficient of the f_3 deformation agrees with the value predicted by the $*$-product then our deformation *must* lead to non-commutative Yang-Mills.

In order to prove that all f_k for $k > 3$ are determined uniquely, it suffices to prove the stronger statement that there exists at most one f_k satisfying (3.9). If given f_j for $j < k$, there exist two distinct functions which can be substituted for f_k in (3.9), then taking their difference gives a function g satisfying

$$f_1(g(x,y),z) + g(f_1(x,y),z) + \text{cyclic} = 0\,. \tag{3.10}$$

This equation is satisfied by $g = f_1$ and $g = f_3$. We will show it has no other solutions. As before, we write

$$g = m^{IJ}\partial_I \wedge \partial_J\,,$$

where $|I| + |J| = 2k$ if m^{IJ} is non zero, and m is antisymmetric. We introduce one more notation. We write the Leibniz rule as follows:

$$\partial_I(BC) = l_I^{JK}\partial_J(B)\partial_K(C)\,,$$

where l_I^{JK} is defined by the Leibniz rule.

With this notation, we expand (3.10):

$$\begin{aligned} &\theta^{ij}m^{IJ}[\partial_i(\partial_I \wedge \partial_J)(x,y)\partial_j(z) + \partial_I(\partial_i(x)\partial_j(y))\partial_J(z) - \partial_I(\partial_j(x)\partial_i(y))\partial_J(z) + \\ &\qquad + \text{cyclic}] = 0\,, \\ \Longrightarrow\ &\theta^{ij}m^{IJ}[\partial_i\partial_I \wedge \partial_J \wedge \partial_j + \partial_I \wedge \partial_i\partial_J \wedge \partial_j + l_I^{ab}\partial_a\partial_i \wedge \partial_b\partial_j \wedge P_J] = 0\,. \end{aligned} \tag{3.11}$$

Now observe that there are many ways of interpreting this "three-form." By evaluating at the origin, it can be construed as a genuine three-form acting on the vector space of polynomials. This allows us to use standard exterior algebra rules for evaluating.

We note that the first 2 terms cancel with the summands in the third term where $|a|$ or $|b| = 0$. This leaves us with

$$\theta^{ij}m^{IJ}\sum_{|a|,|b|>0} l_I^{ab}\partial_a\partial_i \wedge \partial_b\partial_j \wedge \partial_J = 0\,. \tag{3.12}$$

Let us skew diagonalize θ, and suppose that θ^{12} is non zero. Fix a degree $d < 2k$, and consider a ∂_I of degree $|I| = d$ which appears with non-zero m^{IJ} coefficient, some I and the largest number of ∂_1 factors. Call this $I := I_t$. If $d < |I_t|$, then we see that for $l > 1$ there is no term to cancel

$$\theta^{12}m^{I_t,J}\partial_{I_t-\{l\}}\partial_1 \wedge \partial_l\partial_2 \wedge \partial_J$$

unless $|J| = 2$. So, if there exists no $|J| = 2$ factor, $|I_t| = d$. So, we first show that no such terms appear. If there is a degree two factor, say $\partial_J = \partial_s \partial_t$, then we will get terms of the form

$$\theta^{12} m^{JI} r_J^{st} \partial_s \partial_1 \wedge \partial_t \partial_2 \wedge \partial_I = 0 .$$

Once again consider the term $|J| = 2$ with j_1 maximal. Arguing as before (now $|I| > 2$ for $k > 2$), $j_1 = 2$. So, the term

$$2\theta^{12} m^{JI} \partial_1^2 \wedge \partial_1 \partial_2 \wedge \partial_I = 0$$

must be cancelled, but obviously cannot be, so there are no $|J| = 2$ terms.

Returning to our argument for general $|I| > 2$, we obtained a contradiction unless

$$|I_t| = d .$$

We now consider the contribution of the terms $m^{I_t J}$ to (3.12). These give terms,

$$\theta^{12} m^{I_t, J} d \partial_1^d \wedge \partial_1 \partial_2 \wedge \partial_J .$$

It is easy to see that these terms cannot be cancelled by any term with $\partial_I = \partial_1^{d-1} \partial_k$ for that would lead to a ∂_1^d term of the form

$$\theta^{12} m^{I,J} (d-1) \partial_1^d \wedge \partial_k \partial_2 \wedge \partial_J ,$$

which could not cancel for $k \neq 1$. Note the factors of d, $d-1$ above are just those determined by the Leibniz rule.

Therefore, the only possible way to cancel this term is by the contribution from some m^{K,I_t} term. Such a term enters as

$$\theta^{ij} m^{K,I_t} \partial_{K-\{l\}} \partial_i \wedge \partial_l \partial_j \wedge \partial_1^d .$$

For this to cancel the term in question, we need $K - (l \cup i) = J$ and $\partial_l \partial_j = \partial_1 \partial_2$. This implies $K = J$, and gives us two terms which sum to give

$$\theta^{12} m^{J,I_t} (j_1 - j_2) \partial_J \partial_1 \wedge \partial_1 \partial_2 \wedge \partial_1^d ,$$

where again j_i is the exponent of ∂_i in ∂_J.

Hence, for cancellation, $j_2 - j_1 = d$. Suppose we had chosen $d \geq k$. Then we see from this equality that in fact $d = k$ and $j_2 = k$, $j_1 = 0$. We therefore deduce that $m^{IJ} = 0$ unless $|I| = |J| = k$. To simplify further, we must go to the next term in the Leibniz rule expansion. Let J_t be given by $\partial_{J_t} = \partial_2^k$. Consider the term

$$\theta^{12} m^{I_t, J_t} \partial_1^{k-1} \wedge \partial_1^2 \partial_2 \wedge \partial_2^k .$$

As we have shown, only terms with $|I| = |J|$ enter. When $k > 3$, this can only be cancelled against terms $J = J_t$, and $\partial_I = \partial_1^{k-2} \partial_t \partial_s$. In order to match the term $\partial_1^2 \partial_2$ we must have $s = t = 1$. So, there is no new term to cancel against the contribution from $m^{I_t J_t}$. This leads to a contradiction unless g is identically zero. Finally, we can conclude that the f_k are unique up to a scaling of f_3. Since we know that the $*$-product gives one solution for the f_k with $k \geq 3$, it must be the solution determined by supersymmetry.

3.4 Determining B and C

Now that we have shown that the gauge parameter is the one determined by the star product up to a constant, we need to ask whether B^{Λ}_{μ} and C^{Λ}_{a} can always be made to agree with the expected forms,

$$\begin{aligned} (B^{\Lambda}_{\mu})_{ex} &= f(\Lambda, A_{\mu})\,, \\ (C^{\Lambda}_{a})_{ex} &= f(\Lambda, \Psi_{a})\,. \end{aligned} \tag{3.13}$$

With a specific choice of scale for f_3,

$$f(A,B) = A * B - B * A\,,$$

where $*$ denotes the usual star product.

Let us write B and C as perturbations of B_{ex} and C_{ex},

$$\begin{aligned} B^{k} &= B^{k}_{ex} + b^{k}\,, \\ C^{k} &= C^{k}_{ex} + c^{k}\,, \end{aligned} \tag{3.14}$$

where the superscript, k, denotes terms of $\mathrm{O}(\theta^{k})$. We will inductively show that we can choose $b^{k} = 0$ and $c^{k} = 0$. So let us assume that for $j < k$,

$$b^{j} = c^{j} = 0\,.$$

Now closure of the gauge algebra gives the relation,

$$\begin{aligned} [\delta_{\Lambda}, \delta_{\Lambda'}] A_{\mu} &= \delta_{\Lambda} B^{\Lambda'}_{\mu} - \delta_{\Lambda'} B^{\Lambda}_{\mu} \\ &= \delta_{\Lambda_{\theta}} A_{\mu} = \Lambda_{\theta,\mu} + B^{\Lambda_{\theta}}_{\mu}\,. \end{aligned} \tag{3.15}$$

At $\mathrm{O}(\theta^{k})$, this gauge closure relation and its analogue for Ψ yield the conditions:

$$\begin{aligned} \delta^{0}_{\Lambda}(b^{\Lambda'}_{\mu})^{k} - \delta^{0}_{\Lambda'}(b^{\Lambda}_{\mu})^{k} &= 0\,, \\ \delta^{0}_{\Lambda}(c^{\Lambda'}_{a})^{k} - \delta^{0}_{\Lambda'}(c^{\Lambda}_{a})^{k} &= 0\,. \end{aligned} \tag{3.16}$$

What we will show is that (3.16) implies that we can make field redefinitions that eliminate b^{k} and c^{k}, with the possible introduction of new b^{l}, c^{l} with $l > k$. The idea is (3.16) should be viewed as saying that b^{k} and c^{k} are closed one-forms. We then seek an algebraic Poincaré lemma which says that they are in fact exact. By exact, we mean in the sense that there exists $\beta^{k}_{\mu}, h^{k}_{a}$ so that

$$\delta^{0}_{\Lambda}\beta^{k} = (b^{\Lambda})^{k}\,,$$

and

$$\delta^{0}_{\Lambda} h^{k} = (c^{\Lambda})^{k}\,.$$

Then by replacing A_{μ} by,

$$A_{\mu} \longrightarrow A_{\mu} - \beta^{k}_{\mu}\,,$$

and replacing Ψ_{a} by,

$$\Psi_{a} \longrightarrow \Psi_{a} - h^{k}_{a}\,,$$

we remove b^{k} and c^{k} from our gauge transformation.

JHEP03(2002)012

Mimicking the usual Poincaré lemma, we will construct β^k_μ and h^k_a by "integrating" $(b^\Lambda)^k$ and $(c^\Lambda)^k$. We replace derivatives of Λ with appropriate terms in A. Terms with undifferentiated Λ cannot be integrated in this manner and so will have to be handled separately. The procedure to eliminate the terms with Λ derivatives will be the same for c^k and b^k so we treat only the former. We further expand c^k

$$c^k = \sum_p c^{kp} ,$$

where c^{kp} is homogeneous of degree p in A and its derivatives.

Equation (3.16) implies that we can rewrite $\delta_{\Lambda_1}(c^{\Lambda_2})^{kp}$ in the form,

$$\{(\Lambda_1)_{,I}(\Lambda_2)_{,J} + (\Lambda_1)_{,J}(\Lambda_2)_{,I}\}\, c^{kpIJ} .$$

Note that $I, J, \ldots$ are spacetime indices. More generally, the fact that the unperturbed gauge transformation is abelian implies that

$$\delta_{\Lambda_1} \cdots \delta_{\Lambda_{d-1}}(c^{\Lambda_d})^{kp} = (\Lambda_1)_{,I_1} \cdots (\Lambda_d)_{,I_d} c^{kpI_1\cdots I_d} ,$$

with $c^{kpI_1\cdots I_d}$ totally symmetric in the I_j. Since p is finite, there exists a maximal d — call it the depth — so that

$$\delta_{\Lambda_1} \cdots \delta_{\Lambda_{d-1}}(c^{\Lambda_d})^{kp}$$

is non zero, but all higher gauge variations vanish. In particular, for this d, $c^{kpI_1\cdots I_d}$ is gauge invariant.

Let us define

$$h^{kpd} = \frac{1}{d!} A_{I_1} \cdots A_{I_d} c^{kpI_1\cdots I_d} .$$

Then it is easy to see that making the change of variables

$$\Psi_a \longrightarrow \Psi_a - h^{kpd}_a ,$$

has the effect of replacing c^k with a new c^k whose maximal depth is less than d. Performing induction on the depth, we can eventually reduce to the case $d = 0$ which is the same as $c^k = 0$ – at least for all terms in c^k that depend on derivatives of the gauge parameter. A similar argument can be applied to b^k.

We are left with the task of eliminating terms in $(c^\Lambda)^k$ which are of the form Λc^k_0. Namely, terms that do not depend on derivatives of Λ. We return to the condition that the supercharges be gauge invariant,

$$[\delta_\Lambda, \delta_a] A_\mu = 0 ,$$

which implies at $\mathrm{O}(\theta^k)$ that

$$\left(\delta^j_\Lambda \delta^{k-j}_a - \delta^j_a \delta^{k-j}_\Lambda\right) A_\mu = 0 .$$

Assume, inductively, that k is the first order at which the gauge parameters enter undifferentiated. Then the only terms with undifferentiated Λ in the preceding equality are

$$\Lambda(c^k_0)_t(\Gamma_\mu)_{ta} - \delta^0_a \Lambda(b^k_0)_\mu = 0 .$$

Thus, if we can show that $c^k_0 = 0$, we are left with the condition that b^k_0 is a susy invariant and therefore zero. Similarly, the vanishing of $(b^k_0)_\mu$ implies the vanishing of c^k_0.

From the rigidity of the supercharges (that they are gauge-invariant) and from closure of the gauge algebra, we get

$$0 = [\delta_\Lambda, \{\delta_a, \delta_b\}] = [\delta_\Lambda, \Gamma^\nu_{ab}\partial_\nu + \delta_{\Gamma^\nu_{ab}A_\nu} + \delta_v] = [\delta_\Lambda, \delta_{\Gamma^\nu_{ab}A_\nu} + \delta_v]\,. \tag{3.17}$$

Here, δ_v contains all the corrections to the gauge transformation appearing in the closure relation. We expand this relation out (acting on A_μ),

$$\begin{aligned} 0 &= \delta_\Lambda \left\{ \Gamma^\nu_{ab} A_{\nu,\mu} + v_\mu + B_\mu^{(\Gamma^\nu_{ab}A_\nu + v)} \right\} - \delta_{(\Gamma^\nu_{ab}A_\nu + v)} B^\Lambda_\mu \\ &= \delta_{f(\Lambda, \Gamma^\nu_{ab}A_\nu + v)} A_\mu + \delta_\Lambda(\Gamma^\nu_{ab}A_{\nu,\mu} + v_\mu) + B_\mu^{(\delta_\Lambda(\Gamma^\nu_{ab}A_\nu + v))}\,. \end{aligned} \tag{3.18}$$

Collecting the undifferentiated Λ terms at $\mathrm{O}(\theta^k)$ gives

$$0 = (b^k_0)_{\nu,\mu}\,.$$

This in turn implies the vanishing of $(b^k_0)_\nu$ and consequently of c^k_0. By field redefinitions, we can therefore transform B and C into their expected forms.

4. Constraints in commutative variables

As argued in [10], the non-commutative action should be expressible in terms of the commutative variables. We are left with the following puzzle: if the deformation of the gauge symmetry is so strongly constrained by supersymmetry, what is the corresponding constraint in commutative variables? The lagrangian in commutative variables seems to take a fairly generic form with an infinite number of higher derivative terms that depend on F [13] (see, also, [14]),

$$L = F^2 + \theta F^3 + \cdots\,.$$

To address this question, we begin by assuming that

$$\widehat{A} = \widehat{A}(A)\,, \qquad \widehat{\Psi} = \widehat{\Psi}(A, \Psi)\,, \tag{4.1}$$

where the hatted variables are non-commutative variables, while the unhatted variables are commutative. The map between non-commutative and commutative variables can be constructed using the techniques of section 3.4 once the map between $\widehat{\Lambda}$ and Λ is known. In particular, once one has transformed the gauge parameters, eq. (3.16) holds for $B^k_{ex} = 0$. We can then repeat the analysis of that section to inductively construct the commutative variables. At least at the classical level, that construction gives a map from non-commutative to commutative fermions which is linear in the fermions (although non-linear in the gauge bosons).

Using these observations, the change of variables given in [10] can easily be extended to include fermions. To $\mathrm{O}(\theta)$, the change of variables is given by:

$$\begin{aligned} \widehat{A}_\mu &= A_\mu + \frac{1}{2}\theta^{ij} A_i \left(\partial_j A_\mu + F_{j\mu}\right) + \mathrm{O}(\theta^2)\,, \\ \widehat{F}_{\mu\nu} &= F_{\mu\nu} + \theta^{ij}\left(F_{\mu i}F_{j\nu} + A_i \partial_j F_{\mu\nu}\right) + \mathrm{O}(\theta^2)\,, \\ \widehat{\Psi} &= \Psi + \theta^{ij} A_i \partial_j \Psi + \mathrm{O}(\theta^2)\,, \\ \widehat{\Lambda} &= \Lambda + \frac{1}{2}\theta^{ij} A_i \partial_j \Lambda + \mathrm{O}(\theta^2)\,. \end{aligned} \tag{4.2}$$

From the supersymmetry transformations,

$$\begin{aligned}\delta_\epsilon \widehat{A}_\mu &= \frac{i}{2}\,\overline{\epsilon}\,\Gamma_\mu \widehat{\Psi}\,,\\ \delta_\epsilon \widehat{\Psi} &= -\frac{1}{4}\widehat{F}_{\mu\nu}\Gamma^{\mu\nu}\epsilon\,,\end{aligned} \tag{4.3}$$

we can derive transformations for the commutative variables. We learn about N from the $\widehat{A}$ variation,

$$\delta_\epsilon \widehat{A}_\mu = \delta_{A_\nu}\widehat{A}_\mu\ (\delta_\epsilon A_\nu) = \delta_{A_\nu}\widehat{A}_\mu \left(\overline{\epsilon}\left(\frac{i}{2}\Gamma_\nu + N_\nu\right)\Psi\right) = \frac{i}{2}\,\overline{\epsilon}\,\Gamma_\mu\widehat{\Psi}\,,$$

where we have used our usual notation,

$$\delta_\epsilon A_\mu = \overline{\epsilon}\left(\frac{i}{2}\Gamma_\mu + N_\mu\right)\Psi\,.$$

This gives the relation,

$$\begin{aligned}(\Gamma^0 N_\mu\Psi) + \frac{i}{2}\theta^{ij}(\Gamma^0\Gamma_i\Psi)\left(\partial_j A_\mu - \frac{1}{2}\partial_\mu A_j\right) + \frac{i}{2}\theta^{ij}A_i\left(\Gamma^0\Gamma_\mu\partial_j\Psi - \frac{1}{2}\Gamma^0\Gamma_j\partial_\mu\Psi\right) =\\ = \frac{i}{2}\Gamma^0\Gamma_\mu\theta^{ij}A_i\partial_j\Psi\,.\\ \Longrightarrow N_\mu\Psi = \frac{i}{4}\theta^{ij}A_i\left(\Gamma^0\Gamma_j\partial_\mu\Psi\right) - \frac{i}{2}\theta^{ij}(\Gamma^0\Gamma_i\Psi)\left(\partial_j A_\mu - \frac{1}{2}\partial_\mu A_j\right).\end{aligned} \tag{4.4}$$

This teaches us that N is bosonic, and this must be the case to all orders in θ because the transformation between $\widehat{\Psi}$ and Ψ is linear in the fermion field. Note that, curiously, N is *not* gauge invariant.

On the other hand, we learn about M from the $\widehat{\Psi}$ variation which gives the relation,

$$\left(M_{ba} - \frac{1}{4}\theta^{ij}A_i\partial_j F_{\mu\nu}\Gamma^{\mu\nu}_{ba}\right) + \frac{i}{2}\theta^{ij}(\Gamma^0\Gamma_i\Psi)_a\partial_j\Psi_b = -\frac{1}{4}\left\{\theta^{ij}\left(F_{\mu i}F_{j\nu} + A_i\partial_j F_{\mu\nu}\right)\right\}\Gamma^{\mu\nu}_{ba}\,.$$

This leads to the expression,

$$M_{ba} = -\frac{1}{4}\left\{\theta^{ij}F_{\mu i}F_{j\nu}\right\}\Gamma^{\mu\nu}_{ba} - \frac{i}{2}\theta^{ij}(\Gamma^0\Gamma_i\Psi)_a\partial_j\Psi_b\,. \tag{4.5}$$

The special features of this supersymmetry algebra are now visible. The fact that N is bosonic and that the transformation between $\widehat{\Psi}$ and Ψ is linear in Ψ implies that M can never be more than quadratic in fermions. Such an M is morally of the form we should expect for a non-linear sigma model. We can view M as roughly determining a connection. However, the action in commutative variables only contains quadratic couplings in the fermions, again because $\widehat{\Psi}$ is linear in Ψ. This suggests that the "curvature" of the connection (which would be reflected in a four fermion coupling) is zero. This is quite different from what we might have expected from a theory with an infinite number of higher derivative interactions. For example, supersymmetric DBI has many couplings involving large strings of fermions [9].

JHEP03(2002)012

This algebra also violates our constraint (2.12) that the gauge charges be gauge-invariant. Rather,

$$\begin{aligned}[\delta_\Lambda, \delta_\epsilon] A_\mu &= \delta_\Lambda(\epsilon^T \Gamma^0 N_\mu \Psi) \\ &= \partial_\mu(-\frac{i}{4}\theta^{ij}\partial_i\Lambda\, \epsilon^T \Gamma_j \Psi) \\ &= \delta_{\Lambda'} A_\mu \,, \end{aligned} \tag{4.6}$$

where Λ' is a field-dependent gauge transformation. The supersymmetry algebra now takes the form,

$$[\delta_\Lambda, \delta_\epsilon] = \delta_{\Lambda'} \,. \tag{4.7}$$

Note that M is gauge-invariant. We require that the fermion equation of motion be gauge-invariant. However, the fermion equation of motion is needed for closure of the commutator,

$$[\delta_\epsilon, \delta_{\epsilon'}]\Psi \,,$$

on Poincaré. The only place that a problem might arise at $\mathrm{O}(\theta)$ is from

$$\delta_\epsilon F_{\mu\nu}\Gamma^{\mu\nu}\epsilon' - \delta_{\epsilon'} F_{\mu\nu}\Gamma^{\mu\nu}\epsilon \,.$$

What appears at $\mathrm{O}(\theta)$ is the combination,

$$\partial_\mu N_\nu \Psi - \partial_\nu N_\mu \Psi \,.$$

Using (4.6), we see that this is indeed gauge-invariant. This guarantees that the fermion equation of motion is gauge-invariant, as we expect. It seems possible that there is a connection between the non-gauge invariance of δ_ϵ, and the observation of [15] that translations in the non-commutative directions are equivalent to gauge transformations. Whether the special structure of this supersymmetry algebra extends to the $(2,0)$ theory of a six-dimensional self-dual tensor multiplet remains to be seen [11].

Acknowledgments

It is our pleasure to thank N. Itzhaki, H. Liu, and, N. Seiberg for helpful discussions. The work of S.P. is supported in part by NSF Grant No. PHY-0071512, while the work of S.S. is supported in part by NSF CAREER Grant No. PHY-0094328, and by the Alfred P. Sloan Foundation. The work of M.S. is supported in part by NSF Grant No. DMS-9870161.

References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2002) 977 [hep-th/0106048].

[3] S. Paban, S. Sethi and M. Stern, *Constraints from extended supersymmetry in quantum mechanics*, *Nucl. Phys.* **B 534** (1998) 137 [hep-th/9805018]; *Supersymmetry and higher derivative terms in the effective action of Yang-Mills theories*, *J. High Energy Phys.* **06** (1998) 012 [hep-th/9806028].

[4] H. Liu and J. Michelson, $*$*-trek: the one-loop* $N=4$ *noncommutative SYM action*, *Nucl. Phys.* **B 614** (2001) 279 [hep-th/0008205].

[5] H. Liu, $*$*-trek II:* $*_n$ *operations, open Wilson lines and the Seiberg-Witten map*, *Nucl. Phys.* **B 614** (2001) 305 [hep-th/0011125].

[6] M. Pernici, A. Santambrogio and D. Zanon, *The one-loop effective action of noncommutative* $N=4$ *super Yang-Mills is gauge invariant*, *Phys. Lett.* **B 504** (2001) 131 [hep-th/0011140].

[7] R.R. Metsaev and M.A. Rakhmanov, *Fermionic terms in the open superstring effective action*, *Phys. Lett.* **B 193** (1987) 202.

[8] R.R. Metsaev, M.A. Rakhmanov and A.A. Tseytlin, *The Born-Infeld action as the effective action in the open superstring theory*, *Phys. Lett.* **B 193** (1987) 207.

[9] M. Aganagic, C. Popescu and J.H. Schwarz, *Gauge-invariant and gauge-fixed D-brane actions*, *Nucl. Phys.* **B 495** (1997) 99 [hep-th/9612080].

[10] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[11] S. Paban, S. Sethi and M. Stern, under investigation.

[12] M. Cederwall, B.E.W. Nilsson and D. Tsimpis, *Spinorial cohomology and maximally supersymmetric theories*, *J. High Energy Phys.* **02** (2002) 009 [hep-th/0110069].

[13] O.J. Ganor, G. Rajesh and S. Sethi, *Duality and non-commutative gauge theory*, *Phys. Rev.* **D 62** (2000) 125008 [hep-th/0005046].

[14] S.-J. Rey and R. von Unge, *S-duality, noncritical open string and noncommutative gauge theory*, *Phys. Lett.* **B 499** (2001) 215 [hep-th/0007089].

[15] D.J. Gross and N.A. Nekrasov, *Dynamics of strings in noncommutative gauge theory*, *J. High Energy Phys.* **10** (2000) 021 [hep-th/0007204].

JHEP03(2002)012